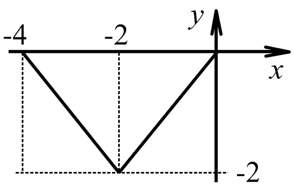
Pismeni ispit iz Analize III, 27.01.2015.
ispit pisati isključivo hemijskom olovkom



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

2. Izračunati $\iint\limits_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R}^2 : y \le x^2+y^2 \le 2y,\ x \ge 0\}$.

3. Izračunati krivoliniski integral $I = \int\limits_c (xy+x+y)dx + (xy+x-y)dy$ ako je $c:\ x^2+y^2 = 3x$.

4. Izračunati površinski integral prvog tipa $I = \iint\limits_W (x^2+y^2)\mathrm{d}S$ gdje je W-površina dijela paraboloida $x^2+y^2 = 2z$ koju odsjeca ravan $z = 1$ (dio paraboloida ispod date ravni).

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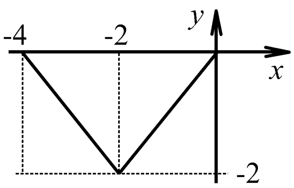

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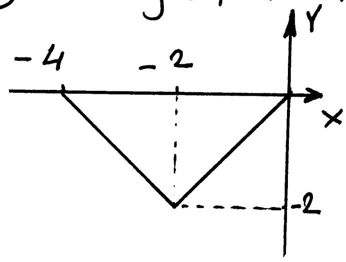


Datu f-ju razviti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Rj: Furijer-ov red za $y=f(x)$ na intervalu (a,b)

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje se koeficijenti a_n, b_n (Furijeovi koeficijenti) računaju po

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \quad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

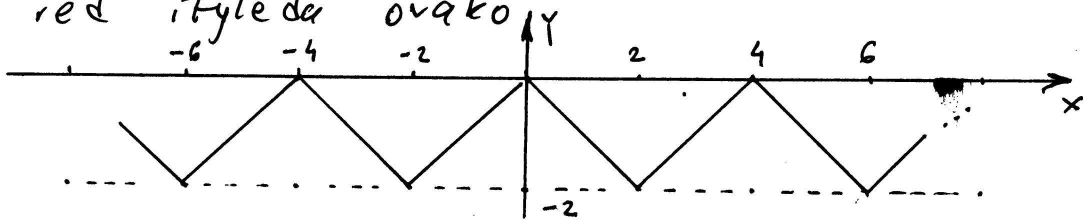
Sad primjetimo da bi f-ju razvili u Furijer-ov red samo po cos-inusima trebamo naštimati da je $b_n=0$, a b_n će biti 0 akko je $f(x)$ parna f-ja a interval simetričan u odnosu na 0.

$$b_n = \frac{2}{b-a}\int_a^b \underbrace{\underbrace{f(x)}_{\text{parna}} \underbrace{\sin\frac{2n\pi x}{b-a}}_{\text{neparna}}}_{\text{neparna}} dx$$

neparna - simetrična u odnosu na koordinatni početak

parna - simetrična u odnosu na y-osu

Drugim riječima treba nam interval $[-4,4]$ a f-ju koju razvijamo u Furijer-ov red izgleda ovako



U stvari primjetimo da (pored intervala $[-4,4]$) možemo posmatrati i interval $[-2,2]$.

$$f(x)=\begin{cases} x, & x\in[-1,0) \\ -x, & x\in[0,1) \end{cases} \qquad [\overset{a}{-2},\overset{b}{2}) \Rightarrow b-a=4, \quad \frac{2n\pi x}{b-a}=\frac{2n\pi x}{4}=\frac{n\pi x}{2}$$

$$\frac{2}{b-a}=\frac{2}{4}=\frac{1}{2}$$

$$a_0=\frac{1}{2}\int_{-2}^{2}\underbrace{f(x)}_{parna}\,dx=\int_0^2 f(x)\,dx=\int_0^2(-x)\,dx=-\frac{1}{2}x^2\Big|_0^2=-2$$

$$a_n=\frac{1}{2}\int_{-2}^{2}\underbrace{\underbrace{f(x)}_{parna}\underbrace{\cos\frac{2n\pi x}{b-a}}_{parna}}_{parna}dx=\frac{1}{2}\cdot 2\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx=\int_0^2(-x)\cos\frac{n\pi x}{2}\,dx$$

$$d\left(\frac{n\pi x}{2}\right)=\frac{n\pi}{2}dx$$

$$=\left|\begin{matrix} u=-x & dv=\cos\frac{n\pi x}{2}dx \\ du=-dx & v=\frac{2}{n\pi}\sin\frac{n\pi x}{2}\end{matrix}\right|=\underbrace{\frac{-2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_0^2}_{0-0}+\frac{2}{n\pi}\int_0^2\sin\frac{n\pi x}{2}\,dx=$$

$$=\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^2=\frac{-4}{n^2\pi^2}(\cos n\pi-1)=\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

Prema tome

$n=1$	(-1)
$n=2$	1
$n=3$	-1

$$f(x)\sim -1+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\frac{n\pi x}{2}=-1+\frac{4}{\pi^2}\sum\frac{2}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{2}$$

$$f(x)\sim -1+\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{(2k-1)\pi x}{2}}{(2k-1)^2}$$ traženi Furijer-ov red

Da je rezultat tačan možemo npr. provjeriti pomoću GeoGebre kucajući naredbu

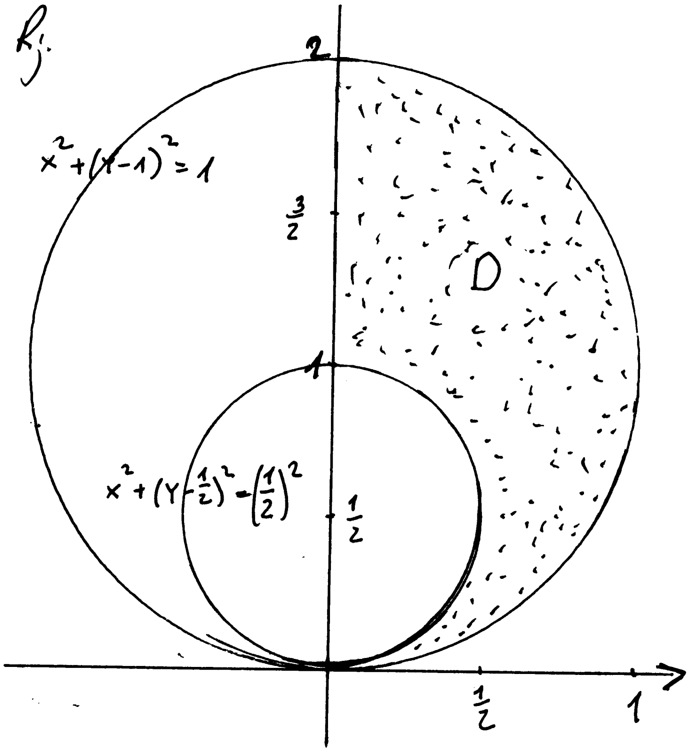
```
y=-1+(8/pi^2)*Sum[Sequence[(cos((2*n-1)*pi*x/2))/(2*n-
-1)^2, n, 1, 100]]
```

Za $x=0 \Rightarrow$ $$0=-1+\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}.$$

Izračunati integral $\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy$ gdje je

$$D=\{(x,y)\in\mathbb{R}^2 \mid y\le x^2+y^2\le 2y,\ x\ge 0\}$$

Rj.



Skicirajmo oblast D.

$$y=x^2+y^2$$

$$x^2+y^2-y=0$$

$$x^2+y^2-2\cdot y\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

$$x^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

$$C\left(0,\frac{1}{2}\right);\ r=\frac{1}{2}$$

$$x^2+y^2=2y$$

$$x^2+y^2-2y=0$$

$$x^2+y^2-2\cdot y\cdot 1+1^2=1^2$$

$$x^2+(y-1)^2=1^2$$

krug sa centrom $C(0;1)$, $r=1$

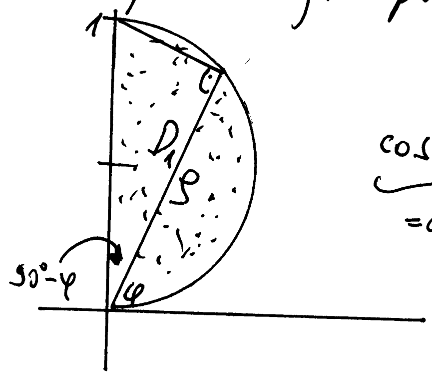
npr. posmatrajmo tačku $\left(\frac{1}{2};\frac{3}{2}\right)$

$$\frac{3}{2}\le \underbrace{\frac{1}{4}+\frac{9}{4}}_{\frac{10}{4}=\frac{5}{2}}\le 3$$

Želimo uvesti polarne koordinate.

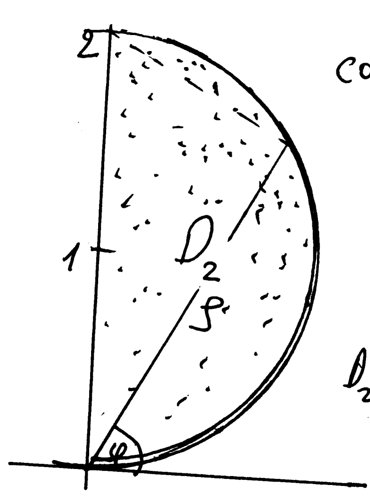
$$x=\rho\cos\varphi$$
$$y=\rho\sin\varphi$$
$$dx\,dy=\rho\,d\rho\,d\varphi$$

Posmatrajmo dvije pomoćne slike



$$\cos(90°-\varphi)=\frac{\rho}{1}$$
$$=\cos 90°\cos\varphi+\sin 90°\sin\varphi=\sin\varphi$$
$$\rho=\sin\varphi$$

$$D_1:\begin{cases}0\le\rho\le\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$



$$\cos(90°-\varphi)=\frac{\rho}{2}$$

$$\rho=2\sin\varphi$$

$$D_2:\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le\rho\le 2\sin\varphi\end{cases}$$

Sad, ako uvedemo polarne koordinate, naša oblast D se transformiše u D' i imamo

$$D\xrightarrow{\text{transformiše}}D'$$

$$D':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ \sin\varphi\le\rho\le 2\sin\varphi\end{cases}$$

$$\iint\limits_{D} \sqrt{x^2+y^2}\,dx\,dy = \left| \begin{array}{l} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{array} \right| = \iint\limits_{D'} \rho\sqrt{\rho^2}\,d\rho\,d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_{\sin\varphi}^{2\sin\varphi} \rho^2\,d\rho = \frac{1}{3}\int_0^{\frac{\pi}{2}} \rho^3 \Big|_{\sin\varphi}^{2\sin\varphi} d\varphi = \frac{7}{3}\int_0^{\frac{\pi}{2}} \underbrace{\cos^3\varphi}_{\cos^2\varphi\cdot\cos\varphi}\,d\varphi =$$

$$= \frac{7}{3}\int_0^{\pi/2} (1-\sin^2\varphi)\,d(\sin\varphi) = \ldots = \frac{14}{9}$$ traženo rješenje

(#) Izračunati krivoliniski integral

$$I = \int_c (xy + x + y)\,dx + (xy + x - y)\,dy \quad \text{ako je } c: x^2 + y^2 = 3x.$$

Rj.

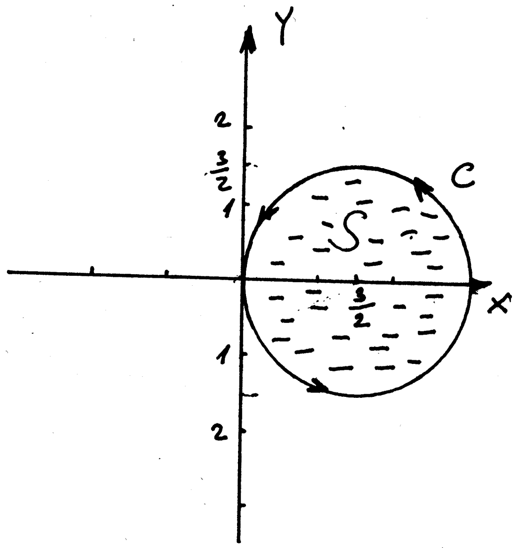
$x^2 + y^2 = 3x$

$x^2 - 3x + y^2 = 0$

$x^2 - 2\cdot x \cdot \frac{3}{2} + \frac{9}{4} - \frac{9}{4} + y^2 = 0$

$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

c: Krug sa centrom u tački $\left(\frac{3}{2}, 0\right)$ poluprečnika $r = \frac{3}{2}$



I način: Greenova formula za ravan

$$\int_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P = xy + x + y, \quad \frac{\partial P}{\partial y} = x + 1, \quad Q = xy + x - y, \quad \frac{\partial Q}{\partial x} = y + 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y + 1 - (x + 1) = y - x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$$x = \frac{3}{2} + r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D: \begin{cases} 0 \le r \le \frac{3}{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I = \int_c (xy + x + y)\,dx + (xy + x - y)\,dy = \iint_S (y - x)\,dx\,dy = \iint_D \left(r\sin\varphi - \left(\frac{3}{2} + r\cos\varphi\right)\right) \cdot r\,dr\,d\varphi$$

$$= \int_0^{3/2} \left[\int_0^{2\pi} \left(r^2\sin\varphi - \frac{3}{2}r - r^2\cos\varphi\right) d\varphi \right] dr = \int_0^{3/2} \Big(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0} - \frac{3r}{2}\varphi\Big|_0^{2\pi} \underbrace{- r\sin\varphi\Big|_0^{2\pi}}_{=0} \Big) dr$$

$$= \int_0^{3/2} -3\pi r\,dr = -3\pi \frac{r^2}{2}\Big|_0^{3/2} = -\frac{3}{2}\pi \cdot \frac{9}{4} = -\frac{27}{8}\pi$$

II način: Klasičan način

C kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$

$$\int_C P(x,y)dx+Q(x,y)dy=\int_a^b [P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_C P(x,y)dx+Q(x,y)dy=\int_{t_1}^{t_2}[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)]dt$$

U našem slučaju c je kružnica. Parametarizirajmo kružnicu

$x=\frac{3}{2}+r\cos\varphi$

$y=r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$x=\frac{3}{2}+\frac{3}{2}\cos t$

$y=\frac{3}{2}\sin t$ gdje $0\le t\le 2\pi$

$\frac{\partial x}{\partial t}=-\frac{3}{2}\sin t$

$\frac{\partial y}{\partial t}=\frac{3}{2}\cos t$

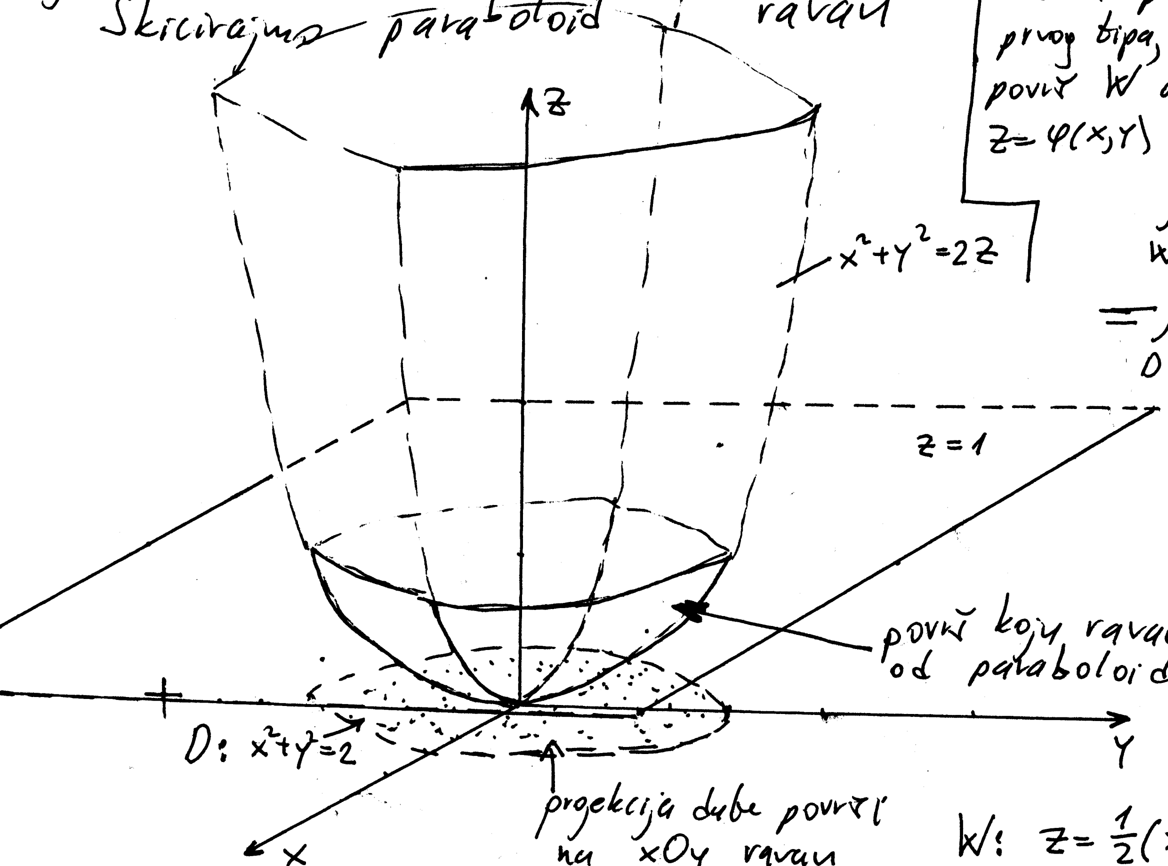
$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy=\int_0^{2\pi}\Big[\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big)+\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)+$$
$$+\big(\tfrac{3}{2}\sin t\big)\Big)\big(-\tfrac{3}{2}\sin t\big)+\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big)+\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)-$$
$$-\big(\tfrac{3}{2}\sin t\big)\Big)\tfrac{3}{2}\cos t\Big]dt=\ldots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I=-\frac{27}{8}\pi$$

(#) Izračunati površinski integral prvog tipa $\iint\limits_W (x^2+y^2)\,dS$, gdje je W – površina dijela paraboloida $x^2+y^2=2z$ koju odsjeca ravan $z=1$ (dio paraboloida ispod date ravni).

Rj. Skicirajmo paraboloid i ravan



Prisjetimo se kako se računa površinski integral prvog tipa, ako je data površ W opisana formulom $z=\varphi(x,y)$

$$\iint\limits_W f(x,y,z)\,dS = \iint\limits_D f(x,y,\varphi(x,y))\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dx\,dy$$

gdje je D projekcija površi W na xOy ravan

$W:\ z=\frac{1}{2}(x^2+y^2)$

$\frac{\partial z}{\partial x}=x,\quad \frac{\partial z}{\partial y}=y$

$$\iint\limits_W (x^2+y^2)\,dS = \iint\limits_D (x^2+y^2)\sqrt{1+x^2+y^2}\,dx\,dy = \left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi \qquad D \xrightarrow{\text{transf.}} D':\begin{cases}0\le r\le\sqrt{2}\\ 0\le\varphi\le 2\pi\end{cases}\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2\end{array}\right.$$

$$=\iint\limits_{D'} r^2\sqrt{1+r^2}\,r\,dr\,d\varphi = \int_0^{2\pi}d\varphi\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,r\,dr = \left|\begin{array}{ll}1+r^2=t^2 & r^2=t^2-1\\ 2r\,dr=2t\,dt & \\ r\,dr=t\,dt & r\Big|_0^{\sqrt{2}} \Rightarrow t\Big|_1^{\sqrt{3}}\end{array}\right|$$

$$=\int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^2-1)\sqrt{t^2}\,t\,dt = \int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}(t^4-t^2)\,dt = \varphi\Big|_0^{2\pi}\left(\frac{1}{5}t^5\Big|_1^{\sqrt{3}}-\frac{1}{3}t^3\Big|_1^{\sqrt{3}}\right)=$$

$$=2\pi\left(\frac{9\sqrt{3}-1}{5}-\frac{3\sqrt{3}-1}{3}\right)=2\pi\,\frac{27\sqrt{3}-3-15\sqrt{3}+5}{15}=2\pi\,\frac{12\sqrt{3}+2}{15}=\frac{(24\sqrt{3}+4)\pi}{15}$$ traženo rješenje